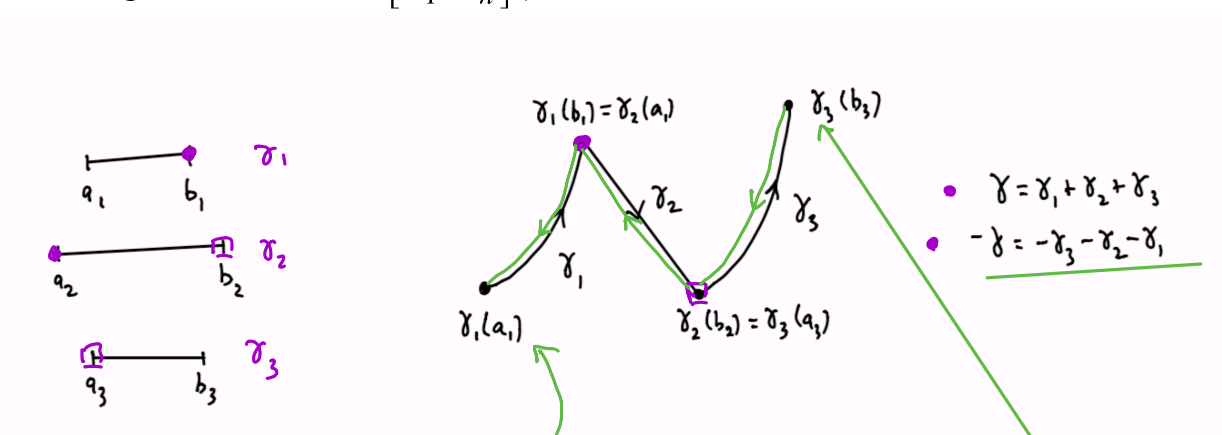
continuing - - -
Friday start with Green

Now, consider piecewise C^1 contours that piece together continuously:

- Def: Let $\gamma_j : [a_j, b_j] \to \mathbb{C}$ be C^1, $j = 1, 2 \ldots, n$. Require
$\gamma_j(b_j) = \gamma_{j+1}(a_{j+1}), j = 1, \ldots n-1$

$$\gamma_j(b_j) = \gamma_{j+1}(a_{j+1}), \qquad j = 1, \ldots n-1.$$

Then $\gamma = \overset{\gamma_1 + \gamma_2 + \ldots + \gamma_n}{\cancel{[\gamma_1, \gamma_2, \ldots \gamma_n]}}$ will be our notation for the piecewise C^1 path obtained from following the paths in order. (The text acually requires that the intervals $[a_j, b_j]$ piece together as well, i.e. $b_j = a_{j+1}$, which we could always accomplish by reparameterizing the curves if necessary. And in that case γ would *actually* be a piecewise C^1 function on the amalgamated interval $[a_1, b_n]$.)



Def For γ as above, define $\gamma_1(a_1)$ to be the *initial point* of γ, and $\gamma_n(b_n)$ to be the *terminal point.*

Def ~~If $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_n]$ is piecewise C^1 as above~~ we write

$$\gamma := \gamma_1 + \gamma_2 + \ldots + \gamma_n$$

and define the contour $-\gamma$ by ~~$-\gamma = [-\gamma_n, -\gamma_{n-1}, \ldots, -\gamma_2, -\gamma_1]$ so~~

$$-\gamma := -\gamma_n - \gamma_{n-1} - \ldots - \gamma_2 - \gamma_1.$$

And we define the contour integral

$$\int_\gamma f(z)\, dz = \int_{\gamma_1 + \gamma_2 + \ldots + \gamma_n} f(z)\, dz := \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz.$$

def of γ

Theorem Let $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$ be a piecewise C^1 curve, with range in $A \subseteq \mathbb{C}$ open. Let $f: A \to \mathbb{C}$ continuous. Then

(1)

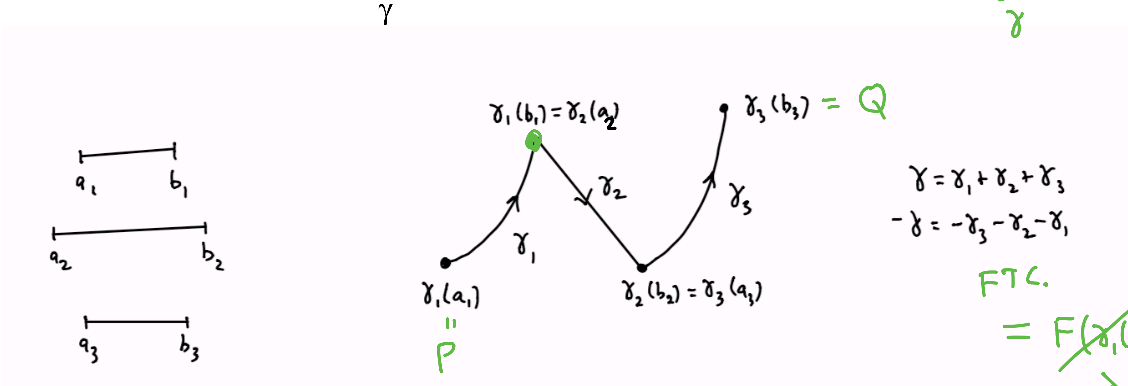
$$\int_{-\gamma} f(z)\, dz = -\int_{\gamma} f(z)\, dz$$

already know

proof: $\boxed{\int_{-\gamma_j} f(z)\, dz = -\int_{\gamma_j} f(z)\, dz.}$ Now sum over j.

(2) If $\exists$ antiderivative $F: A \to \mathbb{C}$ with $F' = f$ then

$$\int_{\gamma} f(z)\, dz = F(Q) - F(P)$$



pf

$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz$$

FTC.

$$= \cancel{F(\gamma_1(b_1))} - F(\gamma_1(a)) + \cancel{F(\gamma_2(b_2))} - \cancel{F(\gamma_2(a_2))} + F(\gamma_3(b_3)) - \cancel{F(\gamma_3(a_3))}$$

$$= F(Q) - F(P)$$

where Q is the terminal point of of γ and P is the initial point.

proof:

$$\int_{\gamma} f(z)\, dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz$$

$$= \sum_{j=1}^{n} F(\gamma_j(b_j)) - F(\gamma_j(a_j))$$

$$= -F(\gamma_1(a_1)) + F(\gamma_1(b_1)) - F(\gamma_2(a_2)) + F(\gamma_2(b_1)) - F(\gamma_3(a_3)) + F(\gamma_3(b_3)) + \ldots - F(\gamma_n(a_n)) + F(\gamma_n(b_n))$$

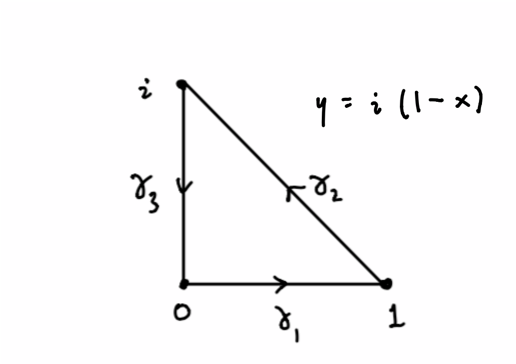
$$F(Q) - F(P)$$

(telescoping series).

estimate

(3) $$\left| \int_{\gamma} f(z)\, dz \right| \underset{\text{def}}{=} \left| \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz \right| \underset{\Delta \text{ ineq.}}{\leq} \sum_{j=1}^{n} \left| \int_{\gamma_j} f(z)\, dz \right| \underset{\text{old estimate}}{\leq} \sum_{j=1}^{n} \int_{\gamma_j} |f(z_j)|\, |dz_j|. := \int_{\gamma} |f(z)|\, |dz|$$

Examples 2: Compute the following. Recall that particular parameterizations don't matter, just the directions of the curves. $\underline{\gamma = \gamma_1 + \gamma_2 + \gamma_3}$.



parameterization | Green's Theorem | FTC

$$\int_\gamma \underbrace{1}\, dz = \qquad z\Big]_0^0 = 0$$

$$\oint_{\partial a} P\,dx + Q\,dy = \iint_a Q_x - P_y \, dA$$

$$\int_\gamma z\, dz = \int_\gamma \underbrace{(x+iy)(dx+idy)} \qquad \frac{z^2}{2}\Big]_0^0 = 0.$$

$$= \int_\gamma (\underbrace{x}_{P}\, dx \underbrace{-y}_{Q}\, dy) + i\int_\gamma \underbrace{y}_{P}\, dx + \underbrace{x}_{Q}\, dy$$

Greens'

$$= \iint_\Delta \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{0}\, dA + i\iint_\Delta \underbrace{1-1}_{0}\, dA = 0$$

$$\int_\gamma \bar{z}\, dz \qquad \text{XXX}$$

$$\int_\gamma (x-iy)(dx+idy) = \int_\gamma \underbrace{x}_{P}\, dx + \underbrace{y}_{Q}\, dy + i\int_\gamma \underbrace{-y}_{P}\, dx + \underbrace{x}_{Q}\, dy$$

$$= \iint_\Delta 0-0\, dA + i\iint_\Delta 2\, dA$$

$$= i \quad !!$$

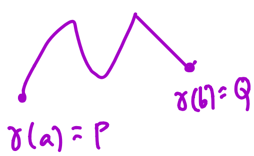
Math 4200-001
Week 5 concepts and homework
2.2-2.3
Due Friday October 2 at 11:59 p.m.

2.2 : 5, 11.
2.3: 1, 3, 5, 7, 9, 10. In 9b write down a homotopy from the given curve to the standard parameterization of the unit circle, in $\mathbb{C}\setminus\{0\}$, to justify your work.

Math 4200
Friday September 25

2.2 Antiderivatives for analytic functions and Cauchy's Theorem: We'll begin by completing Wednesday's notes on contour algebra and the extension of contour integrals to continuous piece-wise C^1 contours γ. In particular we'll check this extension of the FTC:

<u>Theorem</u> (FTC for contour integrals) Let $A \subseteq \mathbb{C}$ open, $f : A \rightarrow \mathbb{C}$ continuous, $\gamma : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{C}$ a piecewise C^1 curve. <u>If</u> f has an analytic antiderivative in A, i.e. $F' = f$, then complex line integrals only depend on the endpoints of the curve γ, via the formula



$$\int_\gamma f(z)\, dz := F(\gamma(b)) - F(\gamma(a))$$

Then the focus of today's notes is to discuss converses to the FTC: namely, what conditions on contour integrals and $f(z)$ imply that $f(z)$ has a complex antiderivative $F(z)$?

Announcements: Hw?

Warm-up exercise:

Contour integrals and antiderivatives:

Let $f: A \subseteq \mathbb{C} \to \mathbb{C}$ continuous, A open and connected. When does f have an antiderivative $F(z)$, i.e. $F'(z) = f(z)\ \forall\ z \in A$? (Note: we've discussed before why antiderivatives on open connected domains are unique up to additive constants, because their differences have zero derivative.)

<u>Theorem 1</u> The following are equivalent, for $f: A \to \mathbb{C}$ continuous, where A is open and connected:

(i) $\exists\ F: A \to \mathbb{C}$ such that $F' = f$ on A

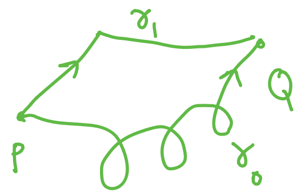
$\Updownarrow$

(ii) Contour integrals are *path independent*, i.e. for all choices of initial point P and terminal point Q in A,

$$\int_{\gamma_0} f(z)\ \mathrm{d}z = \int_{\gamma_1} f(z)\ \mathrm{d}z$$

whenenver γ_0, γ_1 are piecewise C^1 (continuous) paths that start at P and end at Q.

proof:
$(i) \Rightarrow (ii)$ (Use FTC) If γ is any p.w. C^1 contour connecting P to Q in A



and if F exists, $F' = f$ in A

$\Rightarrow$ FTC says $\int_\gamma f(z)dz = F(Q) - F(P)$ ∎

$(ii) \Rightarrow (i)$ We are assuming the following:

(ii) $f: A \to \mathbb{C}$ continuous, where A is open and connected: Contour integrals for f are *path independent*, i.e. for all choices of initial point P and terminal point Q in A,

$$\int_{\gamma_0} f(z)\ dz = \int_{\gamma_1} f(z)\ dz$$

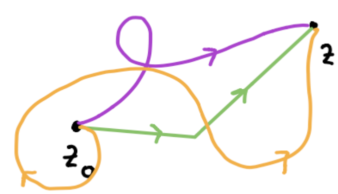
whenenver γ_0, γ_1 are piecewise C^1 paths that start at P and end at Q.

review on Monday 3220 ? $\Rightarrow$

So, fix any $z_0 \in A$. Because A is open and connected it is *pathwise connected*, and for each $z \in A$ there are piecewise C^1 contours in A which start at z_0 and end at z. (See appendix.) Pick any such contour and denote it by $\gamma_{z_0 z}$. Define our candidate antiderivative by

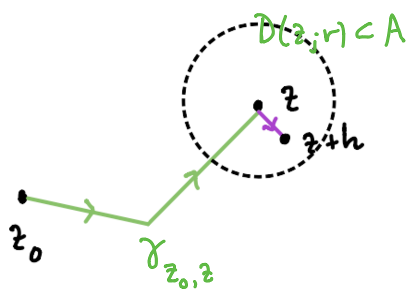
$$\boxed{F(z) := \int_{\gamma_{z_0 z}} f(\zeta)\ d\zeta,}$$

(If there was an antideriv G, then $\int_{\gamma_{z_0,z}} f(\zeta)d\zeta = G(z) - G(z_0)$ so F would be an antideriv too.)



By hypothesis $F(z)$ is well-defined, since contour integrals are path-independent. Our work is to show that F is complex differentiable at each $z \in A$ and that its derivative is f. We'll verify the affine approximation formula for F!

for $z \in A$. Pick $r > 0$ s.t. $D(z;r) \subset A$ open



- $\underbrace{F(z+h)}_{(|h|<r)} \overset{?}{=} F(z) + h f(z) + h\,\varepsilon(h)$ s.t. $\varepsilon(h) \to 0$ as $h \to 0$

$$\| $$

$$\int_{\gamma_{z_0,z}} f(\zeta)d\zeta + \int_{\gamma_{z,z+h}} f(\zeta)d\zeta = F(z) + \int_{\gamma_{z,z+h}} f(\zeta)d\zeta$$

$\gamma_{z_0,z+h} := \gamma_{z_0,z} + \gamma_{z,z+h}$

$\uparrow$ e.g. $\gamma_{z,z+h}(t) = z + th$, $0 \le t \le 1$

$$F(z+h) = F(z) + \int_{\gamma_{z,z+h}} f(z)\,d\zeta + \int_{\gamma_{z,z+h}} f(\zeta) - f(z)\ d\zeta$$

$$F(z+h) = F(z) + f(z)\zeta\Big]_z^{z+h} + h\ \underbrace{\frac{1}{h}\int_{\gamma_{z,z+h}} f(\zeta) - f(z)\,d\zeta}$$

$$F(z+h) = F(z) + f(z)h + h\,\varepsilon(h)$$

$$|\varepsilon(h)| \le \frac{1}{|h|}\int_{\gamma_{z,z+h}} |f(\zeta)-f(z)|\,\overset{\text{arclength}}{|d\zeta|} \le \frac{|h|}{|h|}\underbrace{\max\{|f(\zeta)-f(z)|,\ \text{s.t. } |\zeta-z|<|h|\}}_{\to 0 \text{ as } |h|\to 0}$$

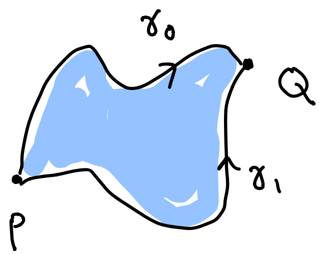
<u>Theorem 2</u> If A is *open* and *simply connected*. Let $f: A \to \mathbb{C}$ be analytic and C^1. Then f has antiderivatives F, unique up to additive constants.

because f is continuous

proof: We'll use Green's Theorem to explain why the path-independence condition (ii) of Theorem 1 holds. Thus antiderivatives exist, and one way to express them is via contour integrals as in the previous discussion:

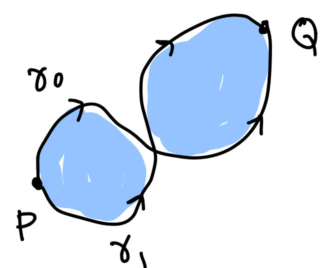
$$F(z) = \int_{\gamma_{z_0 z}} f(\zeta)\,\mathrm{d}\zeta$$

Notice how we will use the "no-holes" idea of *simply-connected*. This explanation is not completely rigorous, but we'll fix that lack of rigor in section 2.3 by defining simply connected more carefully, and also by using different techniques that don't depend on Greens' Theorem and our heuristic pictures of what contours look like.



Cauchy.

$$\oint_{\gamma_1-\gamma_0} f(z)\,dz = 0$$



induct ??

to be continued on Monday!